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Algebraic structure and analytic solutions of generalized three-level Jaynes–Cummings models

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Abstract. A generalized three-level Jaynes–Cummings model (JCM) which includes various ordinary JCMs is shown explicitly to have an $SU(3)$ structure: the Hamiltonian can be treated as a linear function of the generators of the $SU(3)$ group. Based on this algebraic structure, the exact algebraic solutions of the Schrödinger equation, as well as eigenvalues and eigenstates of the Hamiltonian, are obtained by an algebraic method. Thus the three-level JCM is completely solved algebraically. The $SU(N)$ structure of the N-level JCM is also constructed explicitly and can be solved by the same method.

During the last three decades since 1963 when its original form was first proposed [1], the Jaynes–Cummings model (JCM) has been widely used as a full quantum model describing interactions between light and matter [2, 3]. This model, along with many generalized forms, has two apparent advantages. First, the irreducible invariant subspace of the Hilbert space is finite, and it is mathematically soluble. Second, this model exhibits many fascinating quantum effects which can be tested by experiments [4], such as the quantum collapse and revival of atomic inversion [5], squeezing of the radiation field [6] and optical Schrödinger-cat states [7]. The remarkable advance in cavity quantum electrodynamics (QED) experiments involving single atoms (usually Rydberg atoms) within single-mode cavities (the micromaser) [8, 9] and the possibility of finding solutions (often exact) to fundamental models of the quantum theory of interacting field and atoms have excited many efforts to exploit and extend this model. As a result, many generalized forms of JCMs have been proposed. For instance, double-resonance experiments demand more than two levels in the system of interest, because a third level is required to support the second resonance.

It has been noted by many authors that various JCMs exhibit some kinds of similarities [10]. Thus, it is natural to expect that there is a unified description for the solutions of all types of JCMs. It has been shown that all types of two-level JCM have an $SU(2)$ structure [11]: the Hamiltonian can be treated as a linear function of the generators of an $SU(2)$ group, $H = f_0(\Delta) + \sum_{i=0}^{3} f_i(\Delta)X_i$. Where $X_i (i = 1, 2, 3)$ forms a basis of the $SU(2)$ algebra, the operator Δ which commutes with X_i can be treated as a constant in the irreducible representation space of the $SU(2)$ group. Thus any two-level JCM can be mathematically treated as a spin-$\frac{1}{2}$ system in an external magnetic field. From this algebraic structure, it is easy to obtain the evolution matrix, as well as the eigenvalues and eigenstates of the Hamiltonian which are simply algebraic expressions.

It is natural to expect, as pointed out in [11], that an N-level JCM will have an $SU(N)$ structure, as long as there is conservation of excitation which is in general true for most of the generalized JCMs. In the following, we construct this algebraic structure explicitly and, from the viewpoint of algebraic dynamics [12, 13], give a unified description for the eigenvalues and eigenstates of the Hamiltonian, as well as the evolution matrix. When $N = 3$, the solutions are shown to be simply algebraic expressions.

Consider an N-level atom interacting with one mode of electromagnetic field; the Hamiltonian reads

$$H = H_{\mathrm{A}} + H_{\mathrm{F}} + H_{\mathrm{I}} \tag{1}$$

where the free atom part H_{A} and free field part H_{F} are

$$H_{\mathrm{A}} = \sum_{i=1}^{N} \omega_i b_i^+ b_i \qquad H_{\mathrm{F}} = \omega a^+ a + \rho(a^+ a).$$

$\rho(a^+a)$ is usually taken as $\beta a^{+2} a^2$ (the Kerr cavity [14]), but here we treat it as a general real analytic function of a^+a. The atomic levels are labelled according to their energy, the first one being the lowest level, and the Nth level the highest. We have taken $\hbar = 1$ for simplicity, $\omega_i (i = 1, \ldots, N)$ is the ith atomic energy (frequency) and ω is the mode frequency. b_i^+ and b_i are the creation and annihilation operators of an electron at level i, while a^+ and a are those of a photon in the mode. b_i^+ and b_i obey the Fermion commutation rules, and a^+, a obey the Boson commutation rules: $\{b_i, b_j^+\} = \delta_{ij}$, $\{b_i, b_j\} = \{b_i^+, b_j^+\} = 0$; $[a, a^+] = 1$, $[a, b_i] = [a^+, b_i] = 0$. The interaction part H_{I} is usually chosen as one of the following three types:

$$H_I = \sum_{i=2}^{N} \rho_i(a^+a) a^k b_i^+ b_{i-1} + \mathrm{HC} \qquad (\Xi\text{-type})$$

or

$$H_{\mathrm{I}} = \sum_{i=2}^{N} \rho_i(a^+a) a^k b_i^+ b_1 + \mathrm{HC} \qquad (V\text{-type})$$

or

$$H_{\mathrm{I}} = \sum_{i}^{N-1} \rho_i(a^+a) a^{+k} b_i^+ b_N + \mathrm{HC} \qquad (\Lambda\text{-type})$$

where HC means Hermitian conjugate, and $\rho_i(a^+a)$ is the density-dependent coupling coefficients, a^+a is the ordinary number operator. When the integer $k > 1$, the above interactions are usually called density-dependent multiphoton JCMs [10, 11].

The Hamiltonian has two apparent constants of motion: one is the total electron number operator P_{E}, $P_{\mathrm{E}} = \sum_{i=1}^{N} b_i^+ b_i$, and the other is the conservation of excitation Δ:

$$\Delta = a^+a + k\sum_{i=1}^{N} i b_i^+ b_i \qquad (\Xi\text{-type}) \qquad \Delta = a^+a + k\sum_{i=2}^{N} b_i^+ b_i \qquad (V\text{-type})$$

$$\Delta = a^+a - k\sum_{i=1}^{N-1} b_i^+ b_i \qquad (\Lambda\text{-type}).$$

In this paper, we restrict ourselves to the one-electron case: $\sum_{i=1}^{N} b_i^+ b_i = 1$. Thus, the states

$$|\phi(m,i)\rangle = \frac{1}{\sqrt{m!}} a^{+m} b_i^+ |0\rangle \tag{2}$$

form a basis of the Hilbert space, where $|0\rangle$ denotes the no-photon and ground (lowest) atomic state. In this case, the Fermion operators b_i have the following project properties: $b_i b_j = 0$ or $b_i^+ b_j b_k^+ b_l = \delta_{jk} b_i^+ b_l$. Note that the operators $S_{ij} = b_i^+ b_j (i, j = 1, \ldots, N)$ form a basis of the $U(N)$ algebra: $[S_{ij}, S_{kl}] = \delta_{jk} S_{il} - \delta_{il} S_{kj}$. From these commutators, it is easy to show that the operators

$$A_{ij} = \sqrt{\frac{(\Delta - ki)!}{\Delta!} \frac{(\Delta - kj)!}{\Delta!}} a^{ki} a^{+kj} b_i^+ b_j \tag{3}$$

form a basis of the $U(N)$ algebra with the same commutation relations as that of S_{ij},

$$[A_{ij}, A_{kl}] = \delta_{jk} A_{il} - \delta_{il} A_{kj} \tag{4}$$

where $(\Delta - m)!/\Delta! = [\Delta(\Delta - 1)\cdots(\Delta - m + 1)]^{-1}$, and the operator Δ (of Ξ-type) commutes with all members of the algebra $[\Delta, A_{ij}] = 0$. Using the relations

$$a^{+m} a^m = \frac{(a^+ a)!}{(a^+ a - m)!} \qquad a^m a^{+m} = \frac{(a^+ a + m)!}{(a^+ a)!}$$

A_{ij} can also be written as

$$A_{ij} = \sqrt{\frac{(\Delta - kj)!}{(\Delta - ki)!}} a^{+k(j-i)} b_i^+ b_j \qquad A_{ji} = A_{ij}^+ \qquad (i \leqslant j). \tag{5}$$

Taking into account the fact that there is a relation for A_{ii}, $\sum_{i=1}^{N} A_{ii} = \sum_{i=1}^{N} b_i^+ b_i = 1$, the algebra formed by $\{A_{ij}\}$ is indeed $SU(N)$. From expression (5), we can write the Hamiltonian of the N-level JCM (1) of Ξ-type as a linear function of A_{ij},

$$H = \omega\Delta + \sum_{i=1}^{N} \omega_i'(\Delta) A_{ii} + \sum_{i=2}^{N} (f_i(\Delta) A_{i,i-1} + \mathrm{HC}) \tag{6}$$

where $\omega_i'(\Delta) = \omega_i - ki + \rho(\Delta - ki)$, and $f_i(\Delta) = \rho_i(\Delta - ki)\sqrt{\frac{(\Delta-k(i-1))!}{(\Delta-ki)!}}$ which is obtained from the project properties of b_i:

$$\begin{aligned} &\rho_i(a^+ a) a^k b_i^+ b_{i-1} = \rho_i(\Delta - ki) a^k b_i^+ b_{i-1} \\ &\rho(a^+ a) = \sum_{i=1}^{N} \rho(\Delta - ki) b_i^+ b_i. \end{aligned} \tag{7}$$

Since the constant of motion Δ commutes with every member of the $SU(N)$ algebra, it can be treated as a constant in the irreducible representation space of the algebra

$$\Gamma(m) = \{|\phi(m,N)\rangle, |\phi(m+k, N-1)\rangle, \ldots, |\phi(m+(N-1)k, 1)\rangle\} \tag{8}$$

which is also the irreducible invariant subspace of the Hamiltonian, and the state space is the summation of all $\Gamma(m)$.

Similarly, for the V-type or Λ-type interaction, the Hamiltonian can also be treated as linear function of the generators of the $SU(N)$ group. For the V-type, the generators are

$$A_{i1} = \sqrt{\frac{(\Delta - k)!}{\Delta!}} a^k b_i^+ b_1 \qquad A_{1i} = A_{i1}^+ \qquad (i \neq 1) \tag{9}$$

$$A_{11} = b_1^+ b_1 \qquad A_{ij} = b_i^+ b_j \qquad (i, j \neq 1). \tag{10}$$

They have the same commutation relations as that of (5). The Hamiltonian expressed in terms of A_{ij} is

$$H = \omega\Delta + \sum_{i=1}^{N} \omega_i'(\Delta)A_{ii} + \sum_{i=2}^{N}(f_i(\Delta)A_{i1} + \mathrm{HC}) \tag{11}$$

where

$$f_i(\Delta) = \rho_i(\Delta - k)\sqrt{\frac{\Delta!}{(\Delta - k)!}}$$

and

$$\omega_1'(\Delta) = \omega_1 + \rho(\Delta) \text{ and } \omega_i'(\Delta) = \omega_i - k + \rho(\Delta - k).$$

Similarly, for the case of Λ-type, the generators with the same commutation relations are chosen as

$$A_{iN} = \sqrt{\frac{\Delta!}{(\Delta + k)!}}a^{+k}b_i^+b_N \qquad A_{Ni} = A_{iN}^+ \qquad (i \neq N) \tag{12}$$

$$A_{NN} = b_N^+b_N \qquad A_{ij} = b_i^+b_j \qquad (i, j \neq N) \tag{13}$$

and the linear form of the Hamiltonian reads

$$H = \omega\Delta + \sum_{i=1}^{N} \omega_i'(\Delta)A_{ii} + \sum_{i=1}^{N-1}(f_i(\Delta)A_{iN} + \mathrm{HC}) \tag{14}$$

where

$$\omega_i'(\Delta) = \omega_i + k + \rho(\Delta + k) \qquad (i \neq N) \qquad \omega_N'(\Delta) = \omega_N + \rho(\Delta)$$

and

$$f_i(\Delta) = \rho_i(\Delta + k)\sqrt{\frac{(\Delta + k)!}{\Delta!}}.$$

In the above discussions, the key procedures to obtain the algebraic structure are based on the existence of the conservation of excitation Δ and the project properties of $b_i^+b_j$. These two properties enable us to construct an algebra whose members commute with Δ. In fact, in the Hamiltonian (1), for all three types of interaction, there are only $N - 1$ ways of coupling between N atomic levels: for Ξ-type only adjacent levels are coupled, for V-type, the coupling is restricted between the lowest level and other levels, and for Λ-type only the highest level is coupled with other levels. It is easy to see that any kind of N-level JCM has a conservation of excitation in the form $\Delta = a^+a + \sum_i c_i b_i^+ b_i$, if there are only $N - 1$ ways of coupling between N atomic levels where the coefficients c_i is determined by the interaction. As a result, for any N-level JCM, in the one-electron case, the algebraic structure is $SU(N)$ if it has a conservation of excitation.

When $N = 2$, all three kinds of Hamiltonian coincide with each other, and the $SU(2)$ algebraic structure is the same as that of [11]. In this case, the JCM behaves like a spin-$\frac{1}{2}$ system interacting with an external magnetic field; the solutions of the equation of motion can be obtained algebraically in the same way as that of spin-$\frac{1}{2}$ systems. In the following, we show that, when $N > 2$, the solutions of this kind of linear system can also be worked out by an algebraic method.

When a Hamiltonian is expressed as a linear function of a Lie group's generators, there are many algebraic methods to obtain solutions of the equations of motion. One of the methods that can deal with general Lie algebraic structure is the algebraic dynamics

[12, 13]. An important procedure of algebraic dynamics to obtain solutions of a linear system is to find a gauge transformation that transforms the time-dependent Hamiltonian into a linear function of the Cartan operators of the Lie algebra. Then the exact solutions are obtained by the inverse gauge transformation. Since this method involves integrating a set of ordinary differential equations, it cannot be used directly in the Hamiltonians (6), (11) and (14) because the coefficients f_i contain the operator Δ. This difficulty can be overcome if we restrict ourselves in an irreducible representation subspace of the $SU(N)$ algebra in which the operator Δ, and thus the coefficients f_i, can be treated as a constant. The detailed procedure to obtain solutions for a general time-dependent linear system which covers the case of the above discussed JCM can be found in [12, 13].

In the following, we show that, in the case when the considered Hamiltonian is autonomous (it is not dependent on time explicitly), we need only solve algebraic equations to obtain the solutions of the equations of motion, as well as the eigenvalues and eigenstates of the Hamiltonian, if we let the gauge transformation be independent of time. As a result, the solutions for the N-level JCM are algebraic expressions when $N \leqslant 4$. The eigenphases in the time-dependent case can also be obtained in this way.

According to standard Lie algebraic theory [15, 16], if a Hermitian operator is a linear function of the generators of a compact semisimple Lie group, it can be transformed into a linear combination of the Cartan operators of the corresponding Lie algebra by the transformation

$$H \to H' = UHU^{-1} \tag{15}$$

where U is an element of the group which in general has the form

$$U = \prod_{i=1}^{N} \exp(x_i A_i) \tag{16}$$

and $\{A_i\}$ $(i = 1, \ldots, N)$ is a basis set in Cartan standard form of the semisimple Lie algebra, and x_i can be set to zero if the corresponding A_i is a Cartan operator (an element of the Cartan subalgebra). The order of the operators in the above equation can be chosen arbitrarily, but the coefficients x_i are dependent on the order.

From equation (15), the procedure to obtain the solutions of a linear autonomous system is as follows.

(1) Put the expression of U, equation (16), into the right-hand side of (15) and let the coefficients of the non-Cartan operator vanish. Thus one obtains a set of algebraic equations. From these algebraic equations one can obtain a set of solutions of x_i.

(2) From the eigenvalues and eigenstates of H' which are the common eigenvalues and eigenstates of the Cartan operators, one obtains the Hamiltonian's eigenvalues which equal that of H' and the eigenstates by inverse transformation U^{-1}.

(3) The matrix elements of the time evolution operator can be obtained easily from the eigenvalues and eigenstates of the Hamiltonian. Thus one can obtain the solutions of the equations of motion for any initial conditions.

In the first step, we need to assume an order of the operators on the right-hand side of (16). Although any specified order has a solution, a properly chosen order can simplify the procedure to obtain the coefficients x_i. For the N-level JCM with $SU(N)$ structure (whose Cartan operators are $A_{ii} = b_i^+ b_i$), the transformation operator U can be chosen as

$$U = \exp(x_{N1} A_{N1}) \exp(x_{N2} A_{N2}) \ldots \exp(x_{1N} A_{1N}) \tag{17}$$

where the order of the operators of $\exp(x_{ij} A_{ij})(i \neq j)$ is arranged according to the roots of A_{ij} in a decreasing way. For example, the root of A_{N1} is highest, and that of A_{1N} is lowest. With this specification, in our experience the coefficients x_{ij} are relatively easy to work out. Of course, there exist other equally effective choices, especially when one deals with a specified system.

As an illustration, consider the three-level JCM of Ξ-type. Another case can be worked out similarly. Many applications of three-level JCMs can be found, for example, in [17–23]. The Hamiltonian of the three-level JCM of Ξ-type reads

$$H = \omega\Delta + \sum_{i=1}^{3} \omega_i' A_{ii} + f_2 A_{12} + f_3 A_{23} + f_2 A_{21} + f_3 A_{32}. \tag{18}$$

The transformation U is chosen to be six successive transformations

$$U = U_{31} U_{21} U_{32} U_{12} U_{23} U_{13} \tag{19}$$

where $U_{ij} = \exp(x_{ij}(\Delta) A_{ij})$, and the coefficients x_{ij} are determined from the following equations:

$$-\omega' x_{13} - f_2 x_{23} - x_{13}(\omega_3' - f_3 x_{23}) = 0$$

$$f_3 - f_2 x_{13} - \omega_2' x_{23} + \omega_3' x_{23} - f_3 x_{23}^2 = 0 \tag{20}$$

$$f_2 - \omega_1' x_{12} - f_2 x_{12}^2 + f_3 x_{13} + x_{12}(\omega_2' + f_3 x_{23}) = 0 \tag{21}$$

$$f_3 - (\omega_3' - f_3 x_{23}) x_{32} + (\omega_2' - f_2 x_{12} + f_3 x_{23}) x_{32} = 0 \tag{22}$$

$$f_2 + (\omega_1' + f_2 x_{12}) x_{21} - x_{21}(\omega_2' - f_2 x_{12} + f_3 x_{23}) = 0 \tag{23}$$

$$(\omega_1' + f_2 x_{12}) x_{31} - (\omega_3' - f_3 x_{23}) x_{31} + f_2 x_{32} = 0. \tag{24}$$

After the transformation (19), the Hamiltonian becomes

$$H' = \omega\Delta + (\omega_1' + f_2 x_{12}) b_1^+ b_1 + (\omega_2' - f_2 x_{12} + f_3 x_{23}) b_2^+ b_2 + (\omega_3' - f_3 x_{23}) b_3^+ b_3. \tag{25}$$

Note that the solutions of x_{ij} from equations (20)–(24) are algebraic expressions of ω_i' and $f_i (i = 1, 2, 3)$: from equations (20) we obtain the solutions of x_{13} and x_{23}; putting the results into equation (21), we obtain x_{12} by solving a two-order algebraic equation; x_{32}, x_{21}, x_{31} can be obtained in the same way. In fact, equations (20)–(24) are obtained by successively applying the transformation $H \to U_{ij} H U_{ij}^{-1}$, and after each one or two steps we require that the corresponding coefficients of A_{ij} vanish. For example, after the first two transformations U_{13} and U_{23} we obtain equation (20) from the requirement that the coefficients of A_{13} and A_{23} vanish. Then after the third transformation U_{12} equation (21) is obtained by the requirement that the coefficient of A_{12} equals zero. The following equations (22)–(24) are obtained in the same way.

The key of the above procedure is that the vanished generators A_{ij} do not reappear in following steps of transformations. This results from the choice of the order of U_{ij} in equation (19). Indeed, for $SU(N)$ algebra, we have the following equations:

$$\exp(xA_{ij}) A_{ii} \exp(-xA_{ij}) = A_{ii} - xA_{ij}$$

$$\exp(xA_{ij}) A_{jj} \exp(-xA_{ij}) = A_{jj} + xA_{ij}$$

$$\exp(xA_{ij}) A_{ji} \exp(-xA_{ij}) = A_{ji} + x(A_{ii} - A_{jj}) - x^2 A_{ij}$$

$$\exp(xA_{ij}) A_{mi} \exp(-xA_{ij}) = A_{mi} - xA_{mj} \qquad (m \neq i, j)$$

$$\exp(xA_{ij}) A_{jn} \exp(-xA_{ij}) = A_{jn} + xA_{in} \qquad (n \neq i, j)$$

$$\exp(xA_{ij}) A_{mn} \exp(-xA_{ij}) = A_{mn} \qquad (m, n \neq i, j). \tag{26}$$

Thus, if both A_{ij} and A_{mn} have positive (or negative) roots, then the transformation $\exp(xA_{ij})$ transforms A_{mn} into a linear combination of some generators whose absolute values of their roots are equal to or larger than that of A_{mn}, but the sign of the roots remain unchanged. Taking into account the fact that there is an $SU(2)$ subalgebra in the $SU(3)$ algebra, the order of U_{ij} in equation (19), which is arranged according to the corresponding roots, ensures that the vanished roots do not reappear during the following steps of transformations.

From equation (25), it is easy to see that the common eigenstates of H' and Δ are $\{|\phi(m,i)\rangle\}$ of equation (2), and the corresponding eigenvalues $E_{m,i}$ are

$$\begin{aligned}
E_{m,1} &= (m+k)\omega + \omega_1'(m+k) + f_2(m+k)x_{12}(m+k) \\
E_{m,2} &= (m+2k)\omega + \omega_2'(m+2k) - f_2(m+2k)x_{12}(m+2k) + f_3(m+2k)x_{23}(m+2k) \\
E_{m,3} &= (m+3k)\omega + \omega_3'(m+3k) - f_3(m+3k)x_{23}(m+3k).
\end{aligned} \tag{27}$$

$\{E_{m,i}\}$ are also eigenvalues of the Hamiltonian correspond to the eigenstate $|\Psi(m,i)\rangle = U|\phi(m,i)\rangle$ which can be easily obtained from the project properties of A_{ij}: $(A_{ij})^2 = 0$, or $\exp(x_{ij}A_{ij}) = 1 + x_{ij}A_{ij}$. From the Hamiltonian's eigenvalues and eigenstates, we obtain the time evolution matrix

$$U(t) = \sum_{m,i} |\Psi(m,i)\rangle \exp(-\mathrm{i}E_{m,i}t)\langle\Psi(m,i)|. \tag{28}$$

It is block diagonalized in the basis $\{|\phi(m,i)\rangle\}$ and every block is a 3×3 submatrix.

In summary, based on the existence of conservation of excitation and the project property of the Fermion annihilation operators b_i, we construct a unified $SU(N)$ algebraic structure of a generalized N-level JCM. It is the conservation of excitation which leads to the $SU(N)$ structure that makes various JCMs exhibit similarity. Furthermore, the $SU(N)$ structure enables us to describe the solutions of the equation of motion in a unified way.

Although, we restrict ourselves to the one-electron case, the above discussions are equally applicable to the $N-1$ electron case in an N-level JCM. In fact, from the viewpoint of electron–hole duality, the annihilation operator b_i can be viewed as the hole creation operator and b_i^+ the hole annihilation operator $c_i^+ = b_i$, $c_i = b_i^+$. Thus the Hamiltonian in this case can be viewed as a hole interacting with one mode of the field. The total number of holes $\sum_{i=1}^N c_i^+ c_i = 1$ is conserved. This leads to the project property of c_i: $c_i c_j = 0$. Thus, as in the one-electron case, we can construct an $SU(N)$ algebraic structure for this one-hole case. As a result, the two-electron case in the three-level JCM also possesses an $SU(3)$ structure.

The method used to obtain the solutions for the three-level JCM can be similarly used for other cases. In the two-level case, the result is the same as that of [11]. For the four-level case, the solutions can also be written as algebraic expressions. However, when the atomic level $N \geqslant 5$, the general solution cannot be written as an algebraic expression and thus we need to resort to a numeric method to find solutions in an irreducible representation subspace $\Gamma(m)$ of equation (8).

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